On Some Borwein-inspired Properties of

Random Walks with Shrinking Steps

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Introduction. Let $s(x; a_0, a_1, a_2, \dots, a_n)$ denote a product of sinc functions

$$s(x; a_0, a_1, a_2, \dots, a_n) = \prod_{i=0}^{n} \mathrm{sinc}(a_i x) \quad : \quad \text{all } a_i > 0$$

It was, I believe, David Borwein who first noticed that the sequence

$$\begin{aligned}
S_0 &\equiv \int_{-\infty}^{+\infty} s(x; 1) dx = \pi \\
S_1 &\equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}) dx = \pi \\
S_2 &\equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}) dx = \pi \\
S_3 &\equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}) dx = \pi \\
S_4 &\equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9}) dx = \pi \\
S_5 &\equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9}, \tfrac{1}{11}) dx = \pi \\
S_6 &\equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9}, \tfrac{1}{11}, \tfrac{1}{13})) dx = \pi
\end{aligned}$$

does not persist: *Mathematica* supplies

$$\begin{aligned}
S_7 &= \pi \cdot \tfrac{467807924713440738696537864469}{467807924720320453655260875000} \\
&= \pi \cdot 0.9999999999852937 \\
S_8 &= \pi \cdot 0.9999999880796184 < S_7
\end{aligned}$$

Those surprising results were initially attributed to computational error. That and why they are real was explained by David and Jonathan Borwein (father and son) in an intricate paper[1] the simplified essentials of which are presented in a paper by Hanspeter Schmid[2], from whom I now borrow.

The function $\text{sinc}(ax)$ is the Fourier transform of the centered box function of semi-width a and unit area (therefore of height $1/2a$):

$$B(y;a) \equiv \begin{cases} (1/2a) & : \quad -a \leqslant y \leqslant a \\ 0 & : \quad \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{+\infty} B(y;a)e^{-ixy}dy = \frac{1}{2a}\int_{-a}^{+a} e^{-ixy}dy = \frac{\sin(ax)}{ax} \equiv \text{sinc}(ax)$$

Fourier inversion gives

$$B(y;a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{sinc}(ax)e^{ixy}dx$$

This result (note that $\text{sinc}(ax)$ is even) can be rendered

$$2\pi B(y,a) = \int_{-\infty}^{+\infty} \text{sinc}(ax)e^{-iyx}dx$$

which presents the box function

$$\beta(y;a) \equiv 2\pi B(y;a) = \begin{cases} (\pi/a) & : \quad -a \leqslant y \leqslant a \\ 0 & : \quad \text{otherwise} \end{cases}$$

as the direct Fourier transform of $\text{sinc}(ax)$. Setting $y = 0$ we therefore have (in the case $a = 1$)

$$\int_{-\infty}^{+\infty} \text{sinc}(x)dx = \beta(0;1) = \pi$$

To evaluate the Borwein integral of next higher order we use the fact that the Fourier integral of a product is the convolution of the Fourier transforms of the factors. Thus

$$\int_{-\infty}^{+\infty} \text{sinc}(x)\cdot\text{sinc}(\tfrac{1}{3}x)dx = \beta(y,1) * \beta(y,\tfrac{1}{3})\Big|_{y=0}$$

where *Mathematica* supplies

$$\begin{aligned} \beta(y,1) * \beta(y,\tfrac{1}{3}) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \beta(z,1)\beta(z-y,\tfrac{1}{3})dz \\ &= \frac{1}{2\pi}\begin{cases} (4+3y)\pi^2 & : \quad -\frac{4}{3} < y \leqslant -\frac{2}{3} \\ 2\pi^2 & : \quad -\frac{2}{3} < y \leqslant +\frac{2}{3} \\ (4-3y)\pi^2 & : \quad +\frac{2}{3} < y \leqslant +\frac{4}{3} \\ 0 & : \quad \text{elsewhere} \end{cases} \\ &= \pi \text{ at } y = 0 \end{aligned}$$

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[1] "Some remarkable properties of sinc and related integrals," The Ramanujan Journal **6**, 73–89 (2001). The authors acknowledge "useful discussions" with (among others) Richard Crandall.

[2] "Two curious integrals and a graphic proof," Elemente der Mathematik **69**, 11–17 (2014).

which when plotted (FIGURE 4) is seen to have the form of a truncated pyramid (an "eroded" rectangle), with height π and area 2π identical to those of $\beta(y;1)$. "Erosion" occurs because

$$\beta(y;1,\tfrac{1}{3}) \equiv \beta(y,1) * \beta(y,\tfrac{1}{3})$$

reports the result of averaging $\beta(y;1)$ over a sliding base of semi-width $\frac{1}{3}$, which has reduced the width of the π-plateau from 1 to $1-\frac{1}{3}=\frac{2}{3}$. The function $\beta(y;1,\frac{1}{3})$ was seen to be defined segmentally, with junction points at

$$\big\{-1-\tfrac{1}{3},-1+\tfrac{1}{3},+1-\tfrac{1}{3},+1+\tfrac{1}{3}\big\} = \big\{\pm 1 \pm \tfrac{1}{3}\big\}$$

In next higher order we obtain the segmental function

$$\begin{aligned}
\beta(y;1,\tfrac{1}{3},\tfrac{1}{5}) &\equiv \beta(y,1) * \beta(y,\tfrac{1}{3}) * \beta(y,\tfrac{1}{5})\\
&= \frac{1}{2\pi}\begin{cases}
\frac{\pi^2}{60}(529+690y+225y^2) & : \quad -\frac{23}{15}<y<-\frac{17}{15}\\
\pi^2(4+3y) & : \quad -\frac{17}{15}<y<-\frac{13}{15}\\
\frac{\pi^2}{60}(71-210y-225y^2) & : \quad -\frac{13}{15}<y<-\frac{7}{15}\\
2\pi^2 & : \quad -\frac{7}{15}<y<+\frac{7}{15}\\
\frac{\pi^2}{60}(71+210y-225y^2) & : \quad +\frac{7}{15}<y<+\frac{13}{15}\\
\pi^2(4-3y) & : \quad +\frac{13}{15}<y<+\frac{17}{15}\\
\frac{\pi^2}{60}(529-690y+225y^2) & : \quad +\frac{17}{15}<y<+\frac{23}{15}\\
0 \quad : \quad \text{elsewhere}
\end{cases}\\
&= \pi \text{ at } y=0
\end{aligned}$$

with junction points at $\{\pm 1 \pm \frac{1}{3} \pm \frac{1}{5}\}$. So it goes. The junction points of $\beta(y;1,\frac{1}{3},\frac{1}{5},\ldots,\frac{1}{2n+1})$ fall at fall at

$$\big\{\pm 1 \pm \tfrac{1}{3} \pm \tfrac{1}{5} \pm \cdots \pm \tfrac{1}{2n+1}\big\}$$

and the semi-width of the π-plateau is shrinks according to the scheme

$$\begin{aligned}
1 &= \quad 1\\
1-\tfrac{1}{3} &= \quad \tfrac{2}{3}\\
1-\tfrac{1}{3}-\tfrac{1}{5} &= \quad \tfrac{7}{15}\\
1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7} &= \quad \tfrac{34}{105}\\
1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{9} &= \quad \tfrac{67}{315}\\
1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{9}-\tfrac{1}{11} &= \quad \tfrac{422}{3465}\\
1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{9}-\tfrac{1}{11}-\tfrac{1}{13} &= \quad \tfrac{2021}{45045}\\
1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{9}-\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15} &= -\tfrac{982}{45045}
\end{aligned}$$

The occurance of the minus sign signifies that the π-plateau has been eroded away. We have traced the Borwein phenomenon to a property of convoluted box functions, and more specifically to

$$\begin{aligned}
\beta(y;1,\tfrac{1}{3},\tfrac{1}{5},\ldots,\tfrac{1}{2n+1})\Big|_{y=0} &= \pi \quad : \quad n=0,1,2,3,4,5,6\\
&< \pi \quad : \quad n=7,8,9,\ldots
\end{aligned}$$

Borwein & Borwein establish more generally that

$$\begin{aligned} S_\nu \equiv \int_{-\infty}^{+\infty} s(x; a_0, a_1, a_2, \ldots, a_\nu)dx &= \beta(0; a_0, a_1, a_2, \ldots, a_\nu) \\ &= \pi \quad : \quad a_0 - \textstyle\sum_{i=1}^{\nu} a_i \leqslant 0 \qquad (1) \\ &< S_{\nu-1} \quad : \quad \text{otherwise} \end{aligned}$$

Let the sequence $\{a_0, a_1, a_2, \ldots\}$ of positive numbers decrease monotonically, let N be the greatest value of ν for which (1) is satisfied, and let $0 \leqslant n \leqslant N$. Then $\beta(y; a_0, a_1, a_2, \ldots, a_n)$ is formed by splicing distinct segments, with junction points at $a_0 \pm a_1 \pm a_2 \pm \cdots \pm a_n$. It is this fact that provides our point of departure.

Some elementary general principles. Assume $a_i \leqslant a_{i+1}$ $(i = 0, 1, 2, \ldots)$. Then

$$a_0 \pm a_1 \pm a_2 \pm \cdots \pm a_n$$

describes the 2^n possible terminal points $\mathbb{Y}_n = \{y_1, \ldots, y_{2^n}\}$—some of which may be coincident— of a walker who, starting from the origin $(y = 0)$, takes a step of length a_0 to the right, then a step of shorter length a_1 to the right or left, then... until finally a shortest step of length a_n to the right or left. The first-step-to-the-right convention—originally motivated by the Borwein result—is sometimes illuminating, but sometimes not. Its abandonment results in walks of the more symmetric construction

$$\pm a_0 \pm a_1 \pm a_2 \pm \cdots \pm a_n$$

To speak of the *probability* that the walker will, after n steps, stand on a designated element of $\mathbb{Y}_n$ (*i.e.*, to construct a probability density function) it is necessary to assign right/left probabilities to each successive choice point $\pm$. In the one-dimensional literature[3] it is almost invariably assumed (as henceforth in this paper) that the walker always steps right/left with equal probability.

In the theory of random walks in several dimensions one encounters much greater variability in this regard; diverse probability distribution functions are used to determine next-step direction and length. By this means one can, for example, model ordinary and fractional diffusion. Walks in phase space, with next-step probabilities taken from local structure of the Hamiltonian, can—as I have shown elsewhere—be used to construct a "stochastic classical mechanics." Walks on resistive networks, thought of as graphs, with probabilities determined by the conductances of the edges that radiate from each node, can

[3] See, for example, Kent E. Morrison, "Random walks with decreasing steps," available on the web (1998); P. L. Krapivsky & S. Redner, "Random walks with shrinking steps," AJP **72**, 591–598 (2004) and papers cited there. A popular account of the essentials of the K&R paper can be found in §15.2 of Paul Nahin, *Mrs. Perkins's Electric Quilt* (2009).

be used to reproduce Kirchhoff's laws, as was first remarked by S. Kakutani.[4] Some of the topics mentioned above are illustrated in "Some Miscellaneous Adventures in Experimental Mathematical Physics," notes (in the form of a *Mathematica* notebook) of a Reed College physics seminar presented on 9 November 2011. The title was intended to bring to mind the "experimental mathematics" of which Jonathan and Peter Borwein were (in collaboration with—among others—Richard Crandall) founding fathers.

The qualitative features of random walks are most readily exposed by simulation, and it is usually only in the wake of features thus brought to light that one undertakes analytical work. Simulated construction of the endpoints of populations of n-step one-dimensional walks can be produced by the command

$$\texttt{Walks} = \texttt{Table}[\textstyle\sum_{k=0}^{n} \texttt{RandomChoice}[\{\text{-}1,1\}]\,\texttt{a}_k, \{\#\}]$$

where `#` specifies the number of walks to be included in the simulated population. The subsequent command `Union` produces an ordered list of all the distinct endpoints in the sample, but here a cautionary word is in order: the meaning of "distinct" is contingent upon *Mathematica*'s management of round-off errors (as we will have occasion to observe). If the subsequent command `Length` announces 2^n one is assured that the sample was large enough to produce instances of all possible such walks. If, on the other hand, `Length` persists in announcing $\nu < 2^n$ even when `#` is substantially increased one can infer that one has in hand a complete set of walks some members of which have coincident endpoints. The origins of such "co-terminality" will be one of our main concerns.

Authors are most commonly interested in the *distribution* of the endpoints in the sample. To that end, divide the interval bounded by $\pm W$—here

$$W = \sum_{k=0}^{n-1} a_k \text{ sets the bound on an } n\text{-step walk}$$

into (say) 1000 subintervals ("bins") of width $w = \frac{2}{1000}W$ and use the command `BinCounts[Walks,` $\{-W, W, w\}$`]` to distribute the endpoint data among the bins (it is, for this purpose, essential to increase the sample size `#` to a large number). Use `ListPlot` to display the resulting distribution, which when suitably normalized becomes the probability distribution.[5]

Co-terminal harmonic walks. I will be concerned in these pages *not* with the probabilistic distribution of endpoints (though some such distributions will be included among the figures) but (i) with circumstances that give rise to the

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[4] "Markov processes and the Dirichlet problem," Proc. Jap. Acad. **21**, 227–233 (1945); the subject is developed in elaborate detail in Peter G. Doyle & J. Laurie Snell, *Random Walks and Electric Networks* (1984), now available on the web as a free pdf download.

[5] Note that the shape of the distribution depends critically on the bin width. If w is made too small the distribution reduces to an array of discrete spikes.

occurance of n-step walks with **coincident endpoints**, and (ii) with certain **patterns** that emerge when one looks to the intervals that separate next-nearest endpoints. I restrict my attention to walks in which the sequence of step-lengths progress either *harmonically* or *geometrically*.

HARMONIC n-STEP WALKS

Set

$$a_k = \frac{1}{pk+q} \quad : \quad pq \neq 0$$

The resulting sequence $\{a_k\}$ is "harmonic" because each element (after the first) is the harmonic mean (*i.e.*, the reciprocal of the arithmetic mean of the reciprocals) of its nearest neighbors:

$$a_k = \Big[\tfrac{1}{2}\Big(\frac{1}{a_{k-1}} + \frac{1}{a_{k+1}}\Big)\Big]^{-1}$$

From

$$\int_1^n \frac{1}{pk+q}dk = p^{-1}\log\frac{pn+q}{p+q}$$

we infer that all such sequences diverge, if with logarithmic slowness; *i.e.*, that all harmonic n-step walks are unbounded in the limit $n \to \infty$.

UNIFORM n-STEP WALKS Set $p = 0$, $q = 1$. Obtain $\{a_k\} = \{1, 1, 1, \ldots\}$ which gives rise to the simplest of all random walks, one in which every step has the same unit length. After n steps, of which m were to the right and $n-m$ were to the left, the walker stands at $y_m = m - (n-m) = 2m - n$, which is even or odd according as n is. The walker can arrive at y_m in any of binomially many distinct ways; *i.e*, by following any of $\binom{n}{m}$ many distinct co-terminal paths. As n increases the distribution becomes an ever-better approximation to the normal distribution. Such walks model simple one-dimensional diffusion.

BORWEIN n-STEP WALKS Set $p = 2$, $q = -1$. Obtain

$$\{a_{k-1}\}_{k>0} = \big\{1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \ldots\big\}$$

which gives rise to Borwein's shrinking step walks. The set of 2^n n-step Borwein endpoints ranges on a set $\mathbb{Y}_n$ bounded by $\pm W$ with

$$\begin{aligned} W &= 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \cdots + \tfrac{1}{2n-1} \\ &= \tfrac{1}{2}\Big(\text{polygamma}\big[0, n+\tfrac{1}{2}\big] - \text{polygamma}\big[0, \tfrac{1}{2}\big]\Big) \end{aligned}$$

The elements of $\mathbb{Y}_n$ can be determined by a scheme such as the following

$$\left.\begin{aligned}
&+1+\tfrac{1}{3}+\tfrac{1}{5}=+\tfrac{23}{15}=1.533333\\
&-1+\tfrac{1}{3}+\tfrac{1}{5}=-\tfrac{7}{15}\\
&+1-\tfrac{1}{3}+\tfrac{1}{5}=+\tfrac{13}{15}=0.866667\\
&-1-\tfrac{1}{3}+\tfrac{1}{5}=-\tfrac{17}{15}\\
&+1+\tfrac{1}{3}-\tfrac{1}{5}=+\tfrac{17}{15}=1.133333\\
&-1+\tfrac{1}{3}-\tfrac{1}{5}=-\tfrac{13}{15}\\
&+1-\tfrac{1}{3}-\tfrac{1}{5}=+\tfrac{7}{15}=0.466667\\
&-1-\tfrac{1}{3}-\tfrac{1}{5}=-\tfrac{23}{15}
\end{aligned}\quad\right\}\tag{2}$$

(note the sign-reversed bilateral symmetry about the midpoint, the origin of which is obvious), but in higher order it is far easier to proceed by simulation.

Accordingly, I created a population of 1,000,000 simulated 15-step Borwein walks (first step to the right).[6] Members of the population were found to terminate at 16,384 distinct points. But there are $2^{14}=16,384$ such walks altogether. Evidently it is impossible for such walks to be co-terminal.

Generally, if walks $w_{n,1}$ and $w_{n,2}$ are co-terminal siblings (same set of step lengths, different $\pm$ choices) then $A_\mu=\frac{1}{2}(w_{n,1}-w_{n,2})$ must be a μ-step subwalk ($\mu\leqslant n$) that sums to zero (a "null walk"). Given such an A_μ, one has

$$\begin{aligned}
w_{n,1}&=\{+A_\mu,\Omega_{n-\mu}\}\\
w_{n,2}&=\{-A_\mu,\Omega_{n-\mu}\}
\end{aligned}$$

where $\Omega_{n-\mu}$ refers to any n-step walk from which the elements of A_μ have been deleted (of which there are $2^{n-\mu}$).

I was led by my experience in the case $n=15$ to formulate the following

> CONJECTURE : It is impossible from Borwein steps
> to construct such a null walk A_μ.

Ray Mayer promptly supplied a counterexample: he observed that

$$(1+7)-(3+5)=0$$

which when divided by $1\cdot 3\cdot 5\cdot 7=105$ produces

$$\tfrac{1}{105}+\tfrac{1}{15}-\tfrac{1}{35}-\tfrac{1}{21}\equiv A_4=0\tag{3}$$

[6] Such walks are bounded on the right by

$$\begin{aligned}
W=1+\sum_{k=2}^{15}\frac{1}{2k-1}&=\tfrac{1}{2}\Big(\text{polygamma}\big[0,15+\tfrac{1}{2}\big]-\text{polygamma}\big[0,\tfrac{1}{2}\big]\Big)\\
&=2.33587
\end{aligned}$$

and on the left by -0.33587.

But $\frac{1}{105} = \frac{1}{2n-1}$ entails $n = 53$, so it is in the 52-step (first-step-to-the-right) Borwein walk that "Ray's identity" (3) first produces co-terminal walks.[7] There are a total of

$$2^{52} = 4.5036 \times 10^{15}$$

such walks, so to detect the occurance of such co-terminality one would have to simulate something like 10^{18} 52-step walks, which lies beyond the capability of any computer. Yet in that order co-terminality occurs $2^{52-4} = 2.81475 \times 10^{14}$ times, or in 6.25% of the possible 52-step Borwein walks. These remarks illustrate **the hazard latent in experimental mathematics**. And the occasionally surprising power of simple analytical argument.

Ray's argument admits of infinite variation. From (for example[8])

$$1 + 11 = 3 + 9 = 5 + 7$$

we are led to the Borwein null walks

$$\begin{aligned} A_4 &= \tfrac{1}{27} - \tfrac{1}{33} - \tfrac{1}{99} + \tfrac{1}{297} = 0 \\ B_4 &= \tfrac{1}{105} - \tfrac{1}{135} - \tfrac{1}{189} + \tfrac{1}{315} = 0 \\ C_4 &= \tfrac{1}{35} - \tfrac{1}{55} - \tfrac{1}{77} + \tfrac{1}{385} = 0 \end{aligned}$$

Note that the denominators are all odd, and distinct. And that those null walks are disjoint (can occur simultaneously). The implication is that in Borwein walks of order $n \geqslant 384$, of which there are $2^{384} = 3.9402 \times 10^{115}$, we can expect to encounter 8-fold co-terminal walks of the form

$$\{\pm A_4, \pm B_4, \pm C_4, \Omega_{n-12}\}$$

We can, however, expect additional instances of co-terminality to arise similarly from other null walks. It would be difficult to list all the possibilities, and thus to arrive at the total number of distinct endpoints presented by Borwein walks of this or any given order.

$\boxed{\text{SIMPLE HARMONIC } n\text{-STEP WALKS}}$ Set $p = 1$, $q = 0$. Obtain the harmonic sequence

$$\{a_k\} = \left\{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\right\}$$

The resulting simple harmonic n-step walks, if they depart from the origin, are bounded by $\pm H_n = \pm$`HarmonicNumber[n]`. Both H_n and the associated

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[7] We note that, though Borwein walks of ascending order are unbounded, the right bound of such walks (52 steps after a first unit step to the right) lies at 2.96691, not that far beyond the right bound of the 15-step walks considered previously.

[8] Suggested by the *Mathematica* command `Compositions[12,2]`, which produces 2-element partitions of 12. Or use `Partitions[]` to produce a vast assortment of more complex possibilities. Both commands require installation of the Combinatorica package: `Needs["Combinatorica‘"]`.

infinite series are are subjects which, over the centuries, have generated a vast literature.[9] Here I remark only that simple harmonic walks are walks in which co-terminality is ubiquitous. For let $\{a, b, c, d\}$ be any set of distinct (positive or negative) integers that sum to zero:

$$a + b + c + d = 0$$

Division by their product gives

$$\frac{1}{bcd} + \frac{1}{acd} + \frac{1}{abd} + \frac{1}{abc} = 0 \tag{3}$$

where some of the terms will be positive, others negative, and the denominators are distinct. The simplest such construction proceeds from $1+2-3=0$, which gives

$$A_3 = \tfrac{1}{6} + \tfrac{1}{3} - \tfrac{1}{2} = 0$$

and leads to co-terminal n-step harmonic walks for all $n \geqslant 6$. In the case $n = 6$ there are $2^6 = 64$ possible walks, among those occur $2^{6-3} = 8$ co-terminal walks of the form $\{\pm A_3, \Omega_{6-3}\}$. By this reckoning we might expect the set of such walks to possess a total of $2^6 - 2^3 = 56$ distinct endpoints. Simulation indicates, however, that the actual total is 52. Detailed calculation based upon the 64-element harmonic analog of (2) exposes the existence of the harmonic null walk

$$E_4 = 1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{6} = 0$$

which is *not* of the class (3) So among the set of all 6-step simple harmonic walks we encounter $2^{6-4} = 4$ co-terminal walks of the form $\{\pm E_4, \Omega_{6-4}\}$. Those are distinct from the instances of co-terminality produced by A_3, so the number of distinct endpoints becomes $2^6 - 2^3 - 2^2 = 52$, as observed. For 7-step simple harmonic walks the same argument predicts $2^7 - 2^4 - 2^3 = 2(2^6 - 2^3 - 2^2) = 2 \cdot 52 = 104$ distinct endpoints, which is again the number observed.

The case E_4 arises from the circumstance that $6 = 1 + 2 + 3$ is a "perfect number," the sum of its proper divisors (excluding itself). The theory of such numbers has ancient roots, but remains in important respects incomplete. Euclid established—remarkably!— that $q(q+1)/2$ is an even perfect number whenever q is a Mersenne prime (*i.e*, prime of the form $2^p - 1$ with p prime), and Euler proved that all even perfect numbers are of that form. From the primes $2, 3, 5, 7, 13$ we are led thus to the perfect numbers

$$\begin{aligned}
2^1(2^2 - 1) &= 6 \\
2^2(2^3 - 1) &= 28 \\
2^4(2^5 - 1) &= 496 \\
2^6(2^7 - 1) &= 8128 \\
2^{12}(2^{13} - 1) &= 33550336
\end{aligned}$$

[9] See, for example, Julian Havil, *Gamma: Exploring Euler's Constant* (2003).

But not every number of the form $2^p - 1$ is prime ($2^{11} = 23 \cdot 89$). It is not known whether the number of Mersenne primes is finite or infinite (so whether the number of even perfect numbers is finite or infinite), and it is not known whether there exist *any* odd perfect numbers.[10] Using the command `Divisors` to obtain lists of the proper divisors of successive perfect numbers, we obtain (for example)

$$E_6 = 1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{7} - \tfrac{1}{14} - \tfrac{1}{28} = 0$$

which contributes to the co-terminality of n-step simple harmonic walks only for $n \geqslant 28$.

Co-terminal geometric walks. We turn our attention now from walks in which step-length values increment harmonically to walks in which they increment geometrically

$$a_k = \lambda^k \quad : \quad \lambda > 0$$

The sequence $\{\lambda^k\}$ diverges if $\lambda > 1$, and so do the series $\sum_{k=0}^{\infty} \lambda^k$ and the associated random walks $\pm\lambda^0 \pm \lambda^1 \pm \lambda^2 \pm \cdots \pm \lambda^{n-1} : n \to \infty$. We restrict our attention to convergent cases: $\lambda < 1$.

Convergent geometric n-step walks are bounded by $\pm W_n(\lambda)$ with

$$W_n(\lambda) = \frac{1 - \lambda^{n+1}}{1 - \lambda} < \frac{1}{1 - \lambda} = W_\infty(\lambda)$$

The 2^n endpoints of such walks live (if $0 < \lambda < \frac{1}{2}$) within an interval that encloses the interval $[-1, +1\,]$ and is contained within the interval $[-2, +2\,]$; the set $\mathbb{Y}_n$ of such points becomes, as Krapivsky & Redner demonstrate,[3] a Cantor set in the limit $n \to \infty$, with the consequence that the asymptotic distribution of those points cannot be graphed.

Krapivsky & Redner assert also that—by an elementary argument, and as simulations (FIGURE 31) appear to confirm—in the critical case $\lambda = \frac{1}{2}$ the asymptotic distribution is *flat* on the interval $[-2, +2\,]$.

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[10] It is, however, known (as of 2012) that the least odd perfect number—if it exists—is greater than 10^{1500}. Among the many remarkable properties of even perfect numbers I note only that

$$\begin{aligned}
28 &= 1 + 2 + 3 + 4 + 5 + 6 + 7 \\
&= 1^3 + 3^3 \\
496 &= 1 + 2 + 3 + \cdots + 29 + 30 + 31 \\
&= 1^3 + 3^3 + 5^3 + 7^3 \\
8128 &= 1 + 2 + 3 + \cdots + 125 + 126 + 127 \\
&= 1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3 \\
33550336 &= 1 + 2 + 3 + \cdots + 8189 + 8190 + 8191 \\
&= 1^3 + 3^3 + 5^3 + \cdots + 123^3 + 125^3 + 127^3
\end{aligned}$$

So it is with particular interest that we look cases with $\frac{1}{2} < \lambda < 1$. To demonstrate the characteristic features of typical cases, K&R look by simulation to the distribution that results from arbitrarily setting $\lambda = 0.74$. As a first step toward reproduction of their figure, I simulated 500,000 15-step (first-step-to-the-right) geometric walks with λ set to that valu (such walks are bounded on the right by 3.81506, which in the asymptotic limit becomes 3.84615) and found that those walks possessed 16,384 distinct endpoints. But $2^{14} = 16,384$, so none of those 2^{14} geometric walks are co-terminal. On the basis of such evidence I was led to formulate the following

> CONJECTURE: For no distinct $\{p, q, \ldots, r\}$ (and implicitly for no λ) can $\lambda^p \pm \lambda^q \pm \cdots \pm \lambda^r = 0$; *i.e.*, null geometric walks do not exist.

Once again, Ray Mayer promptly supplied a counterexample, of which I provide the simplest instance.

Look to the polynomial $f(\lambda) = \lambda^2 + \lambda - 1$. Immediately $f(0) < 0$ and $f(1) > 1$, so $f(x)$ must possess a real root in the interval $(0, 1)$. That root is in fact given by

$$\lambda = \frac{\sqrt{5} - 1}{2} = \frac{1}{\texttt{GoldenRatio}} = 0.6180339887 \tag{4}$$

which, as it happens, does fall within the "interesting interval" $(\frac{1}{2} < \lambda < 1)$.[11] So *for that particular value of* λ

$$A_3(\lambda) = \lambda^2 + \lambda - 1 = 0$$

provides an instance of a null geometric walk, and so do

$$A_3^{(p)} = \lambda^p(\lambda^2 + \lambda - 1) = 0 \quad : \quad p = 1, 2, 3, \ldots$$

Those introduce instances of co-terminality into geometric n-step walks ($n \geqslant 3$), but those hinge critically upon the precise value (4) of λ, so cannot be expected ever to become evident in casually constructed simulations. We have here again an illustration of **the hazard latent in experimental mathematics**.

Ray's construction admits readily of generalization. We have, for example,

$$A_5^{(p)} = \lambda^p(\lambda^6 + \lambda^5 + \lambda^4 + \lambda - 1) = 0 \quad : \quad \lambda = 0.644524$$

and the list of such examples could be extended indefinitely.

---

[11] Ray himself looked to the next simplest case $g(\lambda) = \lambda^3 + \lambda^2 + \lambda - 1$, which gives

$$\lambda = \frac{1}{3}\Big\{(17 + 3\sqrt{33})^{1/3} - \frac{2}{(17 + 3\sqrt{33})^{1/3}} - 1\Big\} = 0.543689$$

—a number to which one is led also if (looking for advice to the graph of $g(x)$) one commands `FindRoot[g[x]==0,{x,0.5}]`.

Endpoint patterns. My objective here will be simply to describe—and only occasionally to attempt to explain—certain striking patterns that emerge when one looks to ordered sequences $\{e_1, e_2, \ldots, e_\nu\}$ of the distinct endpoints of simulated n-step walks of various kinds. We will look also to the sequences of first differences $d_k = e_{k+1} - e_k$ and to the ordered sequences $\{y_1, y_2, \ldots, y_{2^n}\}$, which will expose patterns of co-terminality. It was graphic experimentation with low-order geometric walks that sparked my interest in this subject, so I look first to some of those.

When 2^n is not too large it becomes feasible to construct an explicit tabulation of the geometric walks of given small order. This is worth doing because—as will emerge—it makes it possible to expose a certain limitation inherent in simulations, and also to discover the specific meanings of certain characteristic results. On the next page I provide a tabulation of the 5-step geometric walks. The table has been designed to permit one to extract tables of 2, 3 and 4-step geometric walks as sub-tables.

5, 6 and 7-step golden walks

Working from the TABLE, with $\lambda = \frac{1}{2}(\sqrt{5}-1)$, one obtains[12] the array of 32 points shown in FIGURE 5. The figure shows 8 instances of double endpoint coincidence and 2 instances of triple coincidence. There are, therefore, only $32-8\cdot1-2\cdot2 = 20$ *distinct* endpoints. To account for the coincidences it is—in view of the bilateral symmetry of the array (the origin of which is obvious)—sufficient to look only to the right half of the figure. The double coincidences all arise from the occurance of null walks $\pm\lambda^{0,1,2}(1-\lambda-\lambda^2)$:

$$
\begin{aligned}
\{1-\lambda-\lambda^2+\lambda^3-\lambda^4, -1+\lambda+\lambda^2+\lambda^3-\lambda^4\} &= \lambda^3-\lambda^4 = 0.09017\\
\{1-\lambda+\lambda^2+\lambda^3-\lambda^4, \quad 1+\lambda-\lambda^2-\lambda^3-\lambda^4\} &= \ \ 1-\lambda^4 = 0.85410\\
\{1-\lambda+\lambda^2+\lambda^3+\lambda^4, \quad 1+\lambda-\lambda^2-\lambda^3+\lambda^4\} &= \ \ 1+\lambda^4 = 1.14590\\
\{1+\lambda-\lambda^2+\lambda^3+\lambda^4, \quad 1+\lambda+\lambda^2-\lambda^3-\lambda^4\} &= \ \ 1+\lambda \ \ = 1.61803
\end{aligned}
$$

So do two of the triple coincidences

$$\{1-\lambda-\lambda^2+\lambda^3+\lambda^4, -1+\lambda+\lambda^2+\lambda^3+\lambda^4\} = \lambda^3+\lambda^4 = 0.38197$$

but the third

$$\{1-\lambda+\lambda^2-\lambda^3-\lambda^4\} = \ \ 1-\lambda \ \ = 0.38197$$

hinges on the identity $1-\lambda = \lambda^3+\lambda^4$, which follows as a *corollary* from the structure of null Golden Walks: $\{1-\lambda-\lambda^2\}+\{\lambda^2-\lambda^3-\lambda^4\} = 0$. The problem of enumerating the various co-terminalities present in n-step Golden Walks is made very difficult by the circumstance that such corollaries exist in unlimited variety.

[12] Use `Sort` to place the endpoints in ascending order.

ENNUMERATION OF ALL POSSIBLE 5-STEP GEOMETRIC WALKS

1. $1+\lambda+\lambda^2+\lambda^3+\lambda^4$
2. $1-\lambda+\lambda^2+\lambda^3+\lambda^4$
3. $1+\lambda-\lambda^2+\lambda^3+\lambda^4$
4. $1-\lambda-\lambda^2+\lambda^3+\lambda^4$
5. $1+\lambda+\lambda^2-\lambda^3+\lambda^4$
6. $1-\lambda+\lambda^2-\lambda^3+\lambda^4$
7. $1+\lambda-\lambda^2-\lambda^3+\lambda^4$
8. $1-\lambda-\lambda^2-\lambda^3+\lambda^4$
9. $1+\lambda+\lambda^2+\lambda^3-\lambda^4$
10. $1-\lambda+\lambda^2+\lambda^3-\lambda^4$
11. $1+\lambda-\lambda^2+\lambda^3-\lambda^4$
12. $1-\lambda-\lambda^2+\lambda^3-\lambda^4$
13. $1+\lambda+\lambda^2-\lambda^3-\lambda^4$
14. $1-\lambda+\lambda^2-\lambda^3-\lambda^4$
15. $1+\lambda-\lambda^2-\lambda^3-\lambda^4$
16. $1-\lambda-\lambda^2-\lambda^3-\lambda^4$
17. $-1+\lambda+\lambda^2+\lambda^3+\lambda^4$
18. $-1-\lambda+\lambda^2+\lambda^3+\lambda^4$
19. $-1+\lambda-\lambda^2+\lambda^3+\lambda^4$
20. $-1-\lambda-\lambda^2+\lambda^3+\lambda^4$
21. $-1+\lambda+\lambda^2-\lambda^3+\lambda^4$
22. $-1-\lambda+\lambda^2-\lambda^3+\lambda^4$
23. $-1+\lambda-\lambda^2-\lambda^3+\lambda^4$
24. $-1-\lambda-\lambda^2-\lambda^3+\lambda^4$
25. $-1+\lambda+\lambda^2+\lambda^3-\lambda^4$
26. $-1-\lambda+\lambda^2+\lambda^3-\lambda^4$
27. $-1+\lambda-\lambda^2+\lambda^3-\lambda^4$
28. $-1-\lambda-\lambda^2+\lambda^3-\lambda^4$
29. $-1+\lambda+\lambda^2-\lambda^3-\lambda^4$
30. $-1-\lambda+\lambda^2-\lambda^3-\lambda^4$
31. $-1+\lambda-\lambda^2-\lambda^3-\lambda^4$
32. $-1-\lambda-\lambda^2-\lambda^3-\lambda^4$

FIGURE 6 was produced by using `Union` to sort and order the endpoints present in a population of 1000 simulated 5-step Golden Walks. It displays only 30 points because `Union` considered the coincident endpoints at ± 1.61803 to be identical, but—because of the way it manages round-off errors—considered all other coincident endpoints to be distinct. This points to a fundamental limitation of the simulation procedure.

The command `DeleteDuplicates[SetPrecision[Data],7]`, with `Data` taken to be that which produced either of the two preceding figures, produces the display of distinct 5-step Golden Walk endpoints shown in FIGURE 7. The display is remarkable for its linearity and seeming regularity. To test regularity we construct a table of first differences, which is plotted in FIGURE 8. We see that all successive differences assume one or the other of only two values. The figure is remarkable also for the symmetry of the pattern in which those values are arrayed; bilateral symmetry was to be expected, but (if one excuses the dangling last point) perfect bilateral symmetry is evident also on each half of the figure.

For the 6 & 7-step siblings of FIGURES 7 & 8 see FIGURES 9 & 10 and FIGURES 11 & 12.[13] *Mathematica* reports a total of 36 distinct endpoints for 6-step Golden Walks, but insists upon including among those four copies of 0. Those arise from $\pm\{1-\lambda-\lambda^2\}\pm\lambda^3\{1-\lambda-\lambda^2\}$. It also reports coincident endpoints ± 2, which arise from $\pm\{1+\lambda+\lambda^2\pm\lambda^3\{1-\lambda-\lambda^2\}\}$ by $\lambda+\lambda^2=1$. The bilateral symmetry of FIGURE 10 has (of course) been retained; that of its individual halves has been lost, but is recovered if one replaces the quadruple 0 by a single 0. Interestingly, the greatest of the 6-step differences was the least of the 5-step differences.

Mathematica reports a total of 54 distinct endpoints for 7-step Golden Walks, and retains no spurious coincidences. Again, the linearity and seeming regularity of the array (FIGURE 11) is conspicuous. The array of first differences (FIGURE 12) is again highly symmetric. The greatest of the 7-step differences was the least of the 6-step differences.

5, 6 AND 7-STEP MAYER WALKS

I turn now to the geometric walks that sparked my interest in this subject, the patterns latent in the null condition $1-\lambda-\lambda^2-\lambda^3=0$, the next higher order sibling of the Golden condition $1-\lambda-\lambda^2=0$. Working again from the list of possible 5-step geometric walks, with λ set now to Ray's value[11] $\lambda=0.543689$, we obtain the array of endpoints shown in FIGURE 13. We see four coincident endpoints. Two of those arise from

$$\pm\{-1+\lambda+\lambda^2+\lambda^3\}+\lambda^4=\lambda^4=0.087378$$

[13] Simulated data suffices in those higher-order cases; explicit 64-line and 128-line step-lists are easy enough to render in *Mathematica* notebooks, but are too large to fit onto the TEX page.

and its negative. The other two arise from

$$1 \pm \lambda\{1 - \lambda - \lambda^2 - \lambda^3\} = 1$$

and its negative. There are evidently $32 - 4 = 28$ distinct endpoints, which are seen in FIGURE 14 to fall in regular linear array. But successive differences range now on a symmetrically distributed set of *three* values (FIGURE 15). Both linearity and 3-point bilateral symmetry persist when one advances from 5-step to 6-step to 7-step Mayer walks (FIGURES 16 & 17). Also persistent is the pattern

least difference becomes greatest difference

when one increments the number of steps.

While linearity was an unanticipated surprise, I am particularly intrigued by several aspects of the difference patterns, and remain unable to account for either.

To place the preceding results in context—to gain a sense of the respect (if any) in which the Golden and Mayer walks (and their higher-order siblings) are "special"—I look to the patterns that develop when λ is assigned Krapivsky & Redner's arbitrarily selected "typical" value $\lambda = 0.74$. Looking specifically to 5-step K&R walks, we find (FIGURE 18) that there are in this case $2^5 = 32$ distinct endpoints (none coincident), and that they present a pattern that —while bilaterally symmetric, and still semi-regular—departs noticably from linearity.[14] The patterned departure from linearity becomes more conspicuous when (FIGURE 19) one looks to the endpoints of 15-step K&R walks. The successive differences derived from 5-step K&R walks (FIGURE 20) fall into a pattern much less orderly than those provided by Golden/Mayer walks, but in which the neighborhoods of two values predominate. The somewhat less "typical" value $\lambda = 1/\sqrt{2} = 0.707107$ produces a more orderly variant of those figures (FIGURES 21 & 22). I conclude that Golden/Mayer walks and their siblings are indeed "special," but for reasons that remain obscure.

5-STEP BORWEIN & SIMPLE HARMONIC WALKS

A generic version of the list on page 13 appears on the next page. If, into that list, one inserts

$$\{a, b, c, d\} = \{\tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9}\}$$

one obtains the 5-step Borwein endpoint data that was used to construct FIGURES 23 & 24. To allude to the nonlinearity of the ordered list of endpoints (FIGURE 23) is to allude simply to the high variability evident in the display (FIGURE 24) of successive differences, which assume $8 = 32/4$ distinct values, the maximal number possible, given the symmetry of the situation. Results derived from 6-step Borwein walks (FIGURES 25 & 26) are qualitatively similar,

[14] The detached endpoints are typical, and arise from the distinction between $\{1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} \pm \lambda^n\}$.

GENERIC 5-STEP WALKS

$$
\begin{array}{rl}
1. & 1+a+b+c+d \\
2. & 1-a+b+c+d \\
3. & 1+a-b+c+d \\
4. & 1-a-b+c+d \\
5. & 1+a+b-c+d \\
 & \vdots \\
29. & -1+a+b-c-d \\
30. & -1-a+b-c-d \\
31. & -1+a-b-c-d \\
32. & -1-a-b-c-d
\end{array}
$$

except that the successive differences assume only 10 distinct values, fewer than the $16 = 64/4$ one might have anticipated. The "least becomes greatest" principle has been lost. The endpoints of 1,000,000 15-step Borwein walks are shown in FIGURE 27 (compare FIGURE 19); the simulation produced a total of $2^{15} = 32768$ distinct endpoints (coincident endpoints not possible for such walks).

Proceeding similarly from

$$\{a, b, c, d\} = \{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}\}$$

one obtains the 5-step simple harmonic endpoint data that was used to construct FIGURES 28 & 29. There are now only 5 distinct successive differences, and they are presented in relatively more orderly array, which shows up as the relatively more perfect linearity of the display of endpoints. The endpoints of 1,000,000 15-step simple harmonic walks are shown in FIGURE 30; the simulation produced only 9664 distinct endpoints (such walks present a high incidence of co-terminality).

These results underscore the exceptional nature of Golden Walks, Mayer Walks and their siblings,[15] and pose this

> PROBLEM : Explain why geometric walks that arise from the conditions $1 - \lambda - \lambda^2 - \cdots - \lambda^n = 0$ possess so few successive differences, why those are patterned as they are, and why they conform to the "least becomes greatest" principle.

[15] Brief allusion to this class of walks appears in Thanu Padmanabhan's *Sleeping Beauties of Theoretical Physics: 26 Surprising Insights* (2015), page 268.

Endpoint distributions for some assorted random walks. The following remarks refer to the production and interpretation of FIGURES 31– 38. In each instance, the bounds $\pm W$ of the walk in question were calculated, and a list of the endpoints of 2,000,000 simulated 20-step walks produced (a task which in most instances took *Mathematica* about two minutes to accomplish). The command `BinCounts` was used to sort the endpoints into 1000 bins of width $2W/1000$. The resulting bin counts were divided by $2W \times 1000$ to produce the "fractional bin count" that is plotted in the figures.

As was remarked on page 10, geometric walks with $0 < \lambda < \frac{1}{2}$ terminate on the points of a Cantor set, so their endpoint distribution "cannot be graphed." Geometric walks with $\frac{1}{2} < \lambda < 1$ are bounded; they give rise to distributions that are "graphable in principle," to—as will emerge—at least the extent that fractal curves are graphable. The "critical case" $\lambda = \frac{1}{2}$ is problematic; it led K&R to devote §4 of their paper[3] to the intricate development of an "exact distribution for $\lambda = 2^{-1/m}$" for which Paul Nahin[3] sketches a much simpler alternative, but is a case which the resourceful Padmanabhan[15] considers to be "extraordinarily hard to analyse." FIGURE 31 supports the assertion that the distribution is in this case flat. Here Bin #0 corresponds to $y = -2$, #1000 corresponds to $y = +2$.

The Golden Walk, which is generated by the polynomial equation

$$1 - \lambda - \lambda^2 = 0$$

and arises from setting $\lambda = \frac{1}{2}(\sqrt{5} - 1) = 1/\text{GoldenRatio}\,\varphi$, gives rise to the distribution approximated by FIGURE 32, which has attracted widespread attention because of its striking self-similar fractal construction. Krapivsky & Redner report[3] that B. Solomyak[16] has used "Golden ratio magic" to account analytically for every detail of the Golden distribution. #s 0 & 1000 correspond to $y = \pm 2.62$, respectively.

The Mayer Walk, which is generated by the polynomial equation

$$1 - \lambda - \lambda^2 - \lambda^3 = 0$$

and arises from setting[11] $\lambda = 0.543689$, gives rise to the self-similar fractal distribution shown in FIGURE 33. #s 0 & 1000 correspond to $y = \pm 2.20$. Note that some of the spikes rise (like δ-functions) beyond 1. This is because here—as always—the form of the simulated curve depends upon bin width. When the number of bins in increased from 1000 to 1100 (and bin width correspondingly decreased) one obtains FIGURE 33B.

K&R's "typical" $\lambda = 0.74$ produces the relatively gentle distribution shown in FIGURE 34, where #s 0 & 1000 correspond to $y = \pm 3.85$. The "less typical" value $\lambda = 1/\sqrt{2} = 0.707107$ produces the curiously structured yet quite benign distribution shown in FIGURE 34: #s 0 & 1000 correspond to $y = \pm 3.42$.

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[16] "On the random series $\sum \pm\lambda^i$ (an Erdös problem)," Ann. Math. **142**, 611–625 (1995).

It will be appreciated that what appear to be "curves" in the preceding figures are actually histograms: $2^{20} = 1,048,576$ points are dropped onto a finite interval that has been partitioned into 1000 subintervals ("bins") and the number of points in each bin displayed. With sufficiently fine resolution the display would consist of a series of δ-spikes, of height determined by co-terminality (the number of walks that share the endpoint in question). But geometric walks with $\frac{1}{2} \leqslant \lambda < 1$ remain bounded even as the number of steps $n \to \infty$, so it becomes meaningful to contemplate the proper curves (which might, in principle, display discontinuities or singularities) that arise in that limit.

But harmonic walks are unbounded in the limit; histograms remain histograms; every such display is specific to a particular value of n, though its qualitative features may persist when the value of n is incremented. FIGURE 36 shows the histogram that results from 20-step Borwein walks, FIGURE 37 shows the quite different histogram that results from 20-step simple harmonic walks. The structure of these histograms is implicit in FIGURES 27 & 30.

Concluding remarks. I came to this subject from the theory of Borwein integrals, and particularly from the convolutions of box functions which account for the "Borwein phenomenon." The historic roots of the theory of geometric walks can, however, be traced to work done eighty years ago by A. Wintner[17] and P. Erdös.[18] P. L. Krapivsky & S. Redner—who were attached to the Center for BioDynamics, the Center for Polymer Studies and the Department of Physics at BostonUniversity—came to it from (or at least were at pains to cite) its physical applications. Curiously, their work (2003) leads them *back to* aspects of the theory of Borwein integrals—FIGURE 4 appears as their Figure 3—but they were unaware of the Borwein paper (2001), and failed to notice the Borwein phenomenon. The theory of convergent geometric walks has been pursued to rarified heights. For example, K&R remark—following Erdös—that the Golden Ratio $\varphi = 1.61803$ is a "Pisot number,"[19] and that the distribution functions that arise from walks with $\lambda = 1/p$ (p a Pisot number $\in (1,2)$) are singular. K&R's report of "little visual evidence of spikiness" for $\lambda \geqslant 0.7$ is supported by FIGURE 34 ($\lambda = 0.74$), but at

$$\lambda = \frac{1}{\text{smallest Pisot number}} = 0.754878$$

[17] B. Jensen & A. Wintner, "Distribution functions and the Riemann zeta function," Trans. Am. Math. Soc. **38**, 48–88 (1935); K. Kershner & A. Wintner, "On symmetric Bernoulli convolutions," Am. J. Math. **57**, 541–548 (1935); A. Wintner, "On convergent Poisson convolutions," *ibid.* **57**, 827–838 (1935).

[18] "On a family of symmetric Berenoulli convolutions," Am. J. Math. **61**, 974–976 (1939); "On smoothness properties of a family of Bernoulli convolutions," **62**, 180–186 (1940).

[19] Often claimed to be the smallest such number, though the real root 1.32471 of $1 + x - x^3 = 0$ was shown by C. L. Segel (1944) to be the actual smallest. A root of a polynomial with integer coefficients is a "Pisot number" if all other roots have modulus less than one.

spikiness is again vividly evident (FIGURE 38), where #s 0 & 1000 correspond to $y = \pm 4.08$.

A word about all this came about. The story has a sad beginning. When I learned of the quite unexpected and untimely death of Richard Crandall (1947–2012), my former student and dear colleague, I immediately relayed news of that sad event to Stephen Wolfram, Jonathan Borwein, Carl Pomerance and Michael Berry, with the first three of whom Richard had frequently collaborated, and with the last of whom—at a conference organized by Borwein—Richard had interacted a few days before his death, and proposed to collaborate. I then found myself in correspondence—touching on a variety of topics—with each of those remarkable individuals.[20]

Joe Buhler—Richard's Reed College classmate and frequent collaborator—returned to the college on 15 September 2016 to present a mathematics seminar. In pre-seminar conversation I mentioned that in Wolfram's recently-published book *Idea Makers* Richard appears in a short list of some very distinguished contributors to the theory and practice of computation (Feynman, Gödel, Turing, von Neumann, Leibniz, Ramanujan, Jobs, ...) but that there is no mention there of the Borwein dynasty (David, Jonathan & Peter: father and sons), which has contributed so conspicuously to the recent development of computer-based "experimental mathematics," and is in some ways reminiscent of the Bernoulli dynasty. Joe responded that they (as Canadians) generally use Maple (instead of *Mathematica*), and mentioned that Johathan Borwein—who was four years younger than Richard—had recently died (on 2 August 2016, at age 65). I went to the web in search of details, and was surprised to discover that—though Borwein was a prolific mathematician, and had with co-authors produced a steady stream of books and papers that report an unusually rich assortment of fascinating results—the Wikipedia article provides links to only two. One presents a series of "Borwein algorithms" that permit decimal approximations to $1/\pi$ to be augmented (not term by term, but) by *factors* of 3 else 4 else 5 else 9. The other is to a paper[1] by Borwein and his father. That 16-page paper proceeds by fairly heavy Fourier analysis, and is not at all transparent. The Wikipedia article "Borwein integral"—taken from §4. Remarks 2 in the original paper—reproduces the most vivid of the results reported by the Borweins, and provides a link to a paper by one Hanspeter Schmid.[2,21] The latter paper employs a graphic argument to provide insight into the origin of the Borweins' surprising result. It was the attractive simplicity

[20] Touching obituaries of Richard by Wolfram and Borwein can be found on the web. Jon Borwein informed me that his Australian grandson, born shortly after Richard's death, bears Richard as his middle name.

[21] Schmid took successive degrees in electrical engineering and information technology from ETH/Zürich (1994–2000) and is presently a professor of analog microelectronics at the University of Applied Sciences in Windisch, Switzerland. From information reported on his website he appears, from the range and variety of his interests (analog circuit design, applied information & probability theory, philosophy of science & technology, music, ... + (evidently) mathematics) he

of Schmid's argument that initially inspired me.

Looking ahead, there are innumerable sequences of interestingly-structued real and natural numbers—the

- Fibonacci numbers $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$,
- triangular numbers $\{1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots\}$,
- pentagonal numbers $\{1, 5, 12, 22, 35, 51, 70, 92, 117, \ldots\}$

etc. come immediately to mind—the reciprocals of which might be used to define the successive step lengths of shrinking walks. A person with an inclination to twiddle (mine is presently exhausted) might look to simulations of some of those. One might look to the endpoint distributions of shrinking walks in several dimensions. More urgent in my own estimation is the PROBLEM posed on page 16. Do patterns such as those to which it refers arise from other/all Pisot walks?

FIGURE CAPTIONS

FIGURE 1: Superimposed graphs of the Borwein integrands $s(x;1), s(x;1,\frac{1}{3})$ and $s(x;1,\frac{1}{3},\frac{1}{5})$.

FIGURE 2: Graphs of the Borwein integrands $s(x;1), s(x;1,\frac{1}{3},\frac{1}{5}\ldots,\frac{1}{11},\frac{1}{13})$ and $s(x;1,\frac{1}{3},\frac{1}{5}\ldots,\frac{1}{13},\frac{1}{15})$. The difference between the latter two is imperceptible, though they bracket the onset of the "Borwein phenomenon."

FIGURE 3: Superimposed Fourier tranforms of the functions $\text{sinc}(x)$, $\text{sinc}(\frac{1}{3}x)$ and $\text{sinc}(\frac{1}{5}x)$.

FIGURE 4: Fourier transforms of $s(x;1), s(x;1,\frac{1}{3}), s(x;1,\frac{1}{3},\frac{1}{5})$, produced by the iterated convolution argument described on pages 2–3.

NOTE: Remarks pertaining to all subsequent figures can be found on pages 12–19.

FIGURE 5: The 2^5 possible endpoints of a 5-step Golden Walk, obtained from the explicit list that appears on page 13. Note that 16 of the walks have doubly coincident endpoints, while 6 have trebly coincident endpoints.

FIGURE 6: `Union` was used to winnow the results of 1000 simulated 5-step Golden Walks. Small decimal differences prevented it from eliminating all but two of the instances of co-terminality (identified by dashed grid lines).

FIGURE 7: `DeleteDuplicates` was used to eliminate all of the duplications present in FIGURE 5. The display is striking for its seeming linearity.

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appears to be a somewhat Crandall-like character himself. In 1999–2000 he published in the Newsletter of the IEEE Professional Communication Society a series of seven essays—based upon an unpublished list of 38 "debate tricks" compiled by Schopenhauer—that are depressingly reminiscent of the quality of recent American political debate (this is, as I write, Election Day 2016).

FIGURE 8: Perfect linearity means perfect invariability of the intervals that separate successive distinct endpoints. The figure displays those differences, shows that all have one or the other of only two (small) values and that they exhibit a striking bilateral symmetry.

FIGURE 9: `DeleteDuplicates` was used in an effort to extract from the data provided by a simulated population of 2000 6-step Golden Walks the set of all distinct endpoints. *Mathematica* considers the four 0s to be distinct (not duplicates), presumably because they differ in high decimal places. The display would be linear if those were (by hand) replaced by a single 0.

FIGURE 10: The associated set of successive differences. The central triple 0 would be eliminated if the quadruple 0 encountered in FIGURE 9 were replaced by a single 0. There would again be only two difference values, again in bisymmetric array. Of those, the largest (0.1803) was for 5-walks the smallest.

FIGURES 11 & 12: These figures, based upon 2000-walk simulations, show that the qualitative features of the preceding constructions pertain also to 7-step Golden Walks. Again the "smallest becomes largest when n is incremented" principle is in evidence.

FIGURES 13, 14 & 15: Working again from the 5-step geometric walk list that appears on page 13, we set λ to Mayer's value $0.543689 < \frac{1}{2}(\sqrt{5}-1) = 0.618034$ and repeat the program that produced FIGURES 5, 7 & 8. The successive differences are seen in FIGURE 15 to assume one or another of *three* values, and to do so in a bisymmetrically patterned array. Bisymmmetry pertains also to the left and right halves of the array.

FIGURE 16: The successive differences pattern for 6-step Mayer walks. The "smallest becomes largest" rule is seen to be operative.

FIGURE 17: The successive differences pattern for 7-step Mayer walks. The "smallest becomes largest" rule is again seen to be operative.

FIGURES 18 & 19: Working again from the 5-step geometric walk list that appears on page 13, we set λ equal to Krapivsky & Redner's "typical" value 0.74 and again repeat the program that produced FIGURES 7 & 8 and see that *linearity is now lost*; we see that there are now 8 distinct difference values. Central bisymmetry is maintained, but that of the right/left halves of the figure is lost. The first and last points stand well apart from the others.

FIGURE 20: Display of the distinct endpoints of a simulated population of 1,000,000 15-step K&R walks. Nonlinearity is obvious.

FIGURES 21 & 22: Working again from the 5-step geometric walk list that appears on page 13, we set λ to the arbitrary but somewhat "less typical" value $2^{-\frac{1}{2}} = 0.707107$ and repeat the program that produced FIGURES 18 & 19. Linearity is lost, but less vividly. There are now only 5 distinct difference values, and their bisymmetric arrangement is more orderly.

Figures 23 & 24: In the generic 5-step list (page 16) assign to parameters $\{a, b, c, d\}$ the Borwein harmonic values $\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\}$. The resulting array of 32 distinct endpoints is highly nonlinear, the 8 successive difference values widely scattered.

Figures 25 & 26: Similar remarks pertain to the data produced by simulation of 6-step Borwein walks. Again, the bilateral symmetry evident in the right/left halves of the differences figure is striking.

Figure 27: Display of the distinct endpoints of a simulated population of 1,000,000 15-step Borwein walks.

Figures 28 & 29: In the generic 5-step list (page 16) assign to parameters $\{a, b, c, d\}$ the simple harmonic values $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$. The resulting array of 32 distinct endpoints is more nearly linear than that produced by the 5-step Borwein walks, the $5 < 8$ successive difference values—though still widely scattered (on a smaller interval)—more neatly arranged.

Figure 30: Display of the distinct endpoints of a simulated population of 1,000,000 15-step simple harmonic walks.

Figure 31: The flat distribution produced when the endpoints of 2,000,000 simulated 15-step geometric walks (λ set to the critical value $\frac{1}{2}$) are sorted into 1000 bins of equal width. See page 17 for details.

Figure 32: The self-similar fractal distribution produced when the endpoints of 2,000,000 simulated 20-step Golden Walks ($\lambda = \frac{1}{2}(\sqrt{5} - 1) = 0.618034$) are sorted into 1000 bins of equal width.

Figure 33: The self-similar fractal distribution produced when the endpoints of 2,000,000 simulated 20-step Mayer Walks ($\lambda = 0.543689 < \frac{1}{2}(\sqrt{5} - 1)$) are sorted into 1000 bins of equal width. Some spikes rise (like δ-functions)above 1.

Figure 33B: Endpoints of the same population of Mayer walks, but with bin width reduced (number of bins increased to 1100). Demonstrates that all such figures are artifacts of the selected bin width.

Figure 34: The distribution that results when the endpoints of 2,000,000 simulated 20-step K&R walks ($\lambda = 0.74$) are sorted into 1000 bins of equal width.

Figure 35: The distribution that results when the endpoints of 2,000,000 simulated 20-step geometric walks with $\lambda = 1/\sqrt{2} = 0.707107$ are sorted into 1000 bins of equal width.

Figure 36: The distribution that results when the endpoints of 2,000,000 simulated 20-step Borwein walks are sorted into 1000 bins of equal width. The central dip reflects a feature evident in Figure 27.

Figure 37: The distribution that results when the endpoints of 2,000,000 simulated 20-step simple harmonic walks are sorted into 1000 bins of equal width. Comparison of Figure 30 with Figure 27 provides indication of why the "central dip" has disappeared.

FIGURE 38: The (presumably) self-similar fractal distribution produced when the endpoints of 2,000,000 simulated 20-step geometric walks with

$$\lambda = \frac{1}{\text{smallest Pisot number}} = 0.754878 > \frac{1}{\text{golden ratio}} = 0.618034$$

are sorted into 1000 bins of equal width. The distribution appears to be a spiky sibling of the K&R distribution ($\lambda = 0.74$) shown in FIGURE 34.

FIGURE 39: Working again from the 5-step geometric walk list that appears on page 13, we set λ to Pisot's value and obtain an array of $24 < 2^5$ distinct endpoints that is less nearly linear than one might have anticipated. The bounding endpoints stand apart, in a way we have seen to be characteristic of all n-step walks.

FIGURE 40: The associated array of successive differences exhibits central bilateral symmetry (but not such symmetry on its left/right halves), and the differences assume 6 distinct values—more than our experience with Golden and Mayer walks might lead us to anticipate.